Coloring invariant. Knot arithmetic

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Reidemeister moves

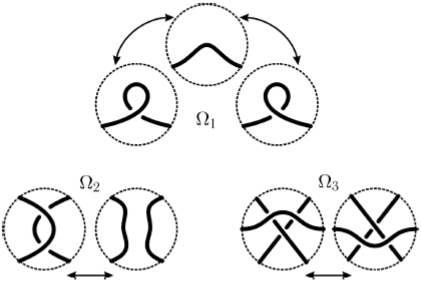

Figure : Reidemeister moves

Theorem

Two knots are equivalent **if and only if** *a diagram of the first knot can be transformed into a diagram of the second knot by a finite sequence of Reidemeister moves and trivial moves.*

Knot invariants

Definition

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- Coloring invariant $col_3(D)$ is the number of correct colorings.

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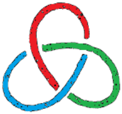
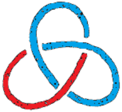


Correct



Incorrect

Coloring invariant is an invariant

Theorem. The number of good colorings of a knot diagram is a knot invariant.

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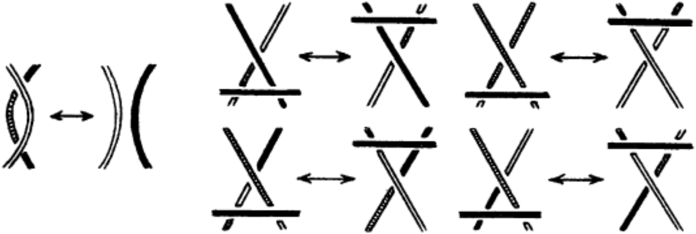

Figure : Invariance under Reidemeister moves

Exercises

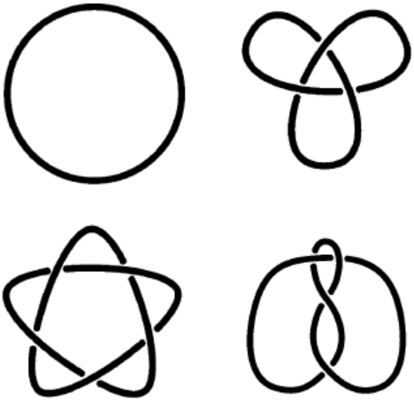

Figure : Find the coloring invariant of these knots

Application: naked elastic human wearing a watch

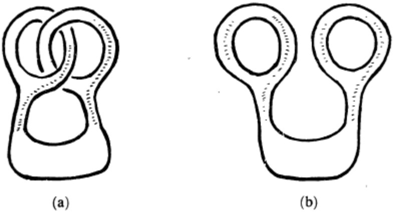

Figure : Recall: elastic human

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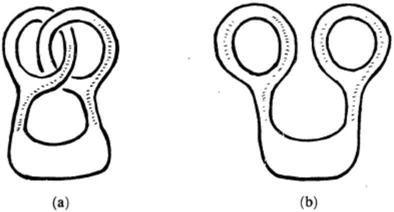

Figure : Recall: elastic human

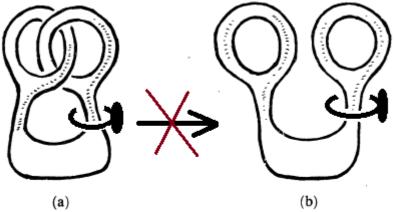

Figure : Elastic human wearing a watch

n-coloring

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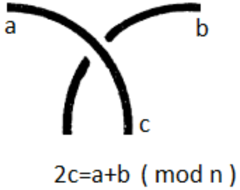
Coloring invariant $col_n(D)$ is the number of correct colorings.



Can we distinguish figure 8 knot and trefoil?

Figure 8 knot and trefoil

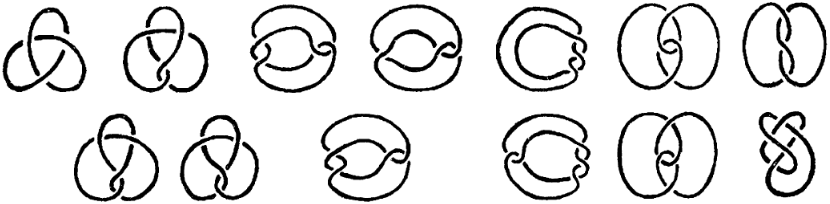


Figure : Figure 8 knot and trefoil

First method

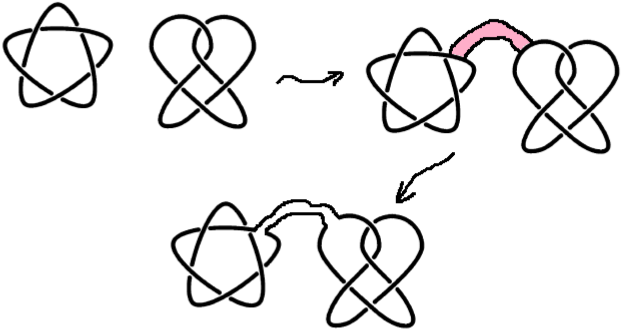
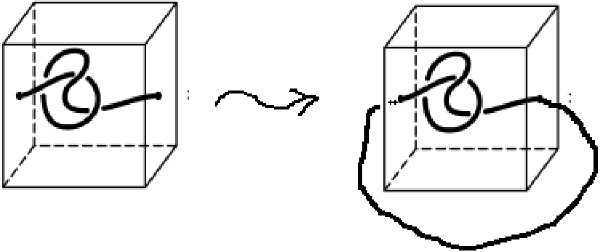


Figure : Connected product of two knots

Box description of a knot



Second method

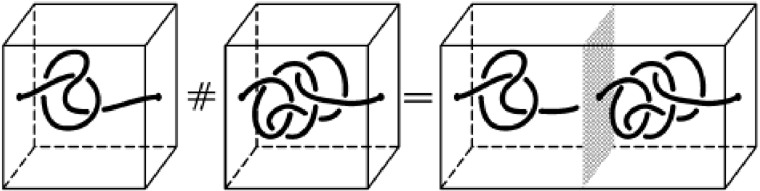


Figure : Connected product in box presentation

Properties

Lemma

The operation $\#$ is associative and commutative, and the unknot U is the identity element. In other words, for any three knots A, B, C, we have $(A\#B)\#C = A\#(B\#C)$, $A\#B = B\#A$ and $U\#A = A\#U = A$.

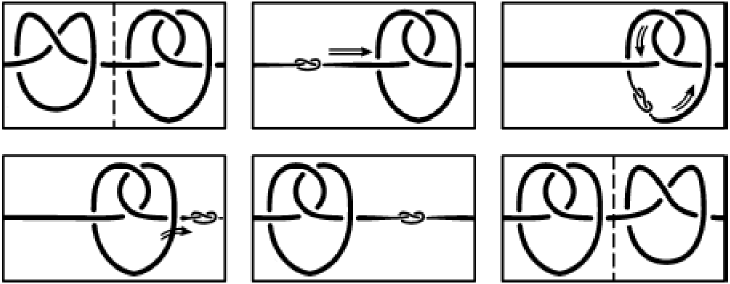


Figure : Commutativity of composition